Hitting time theorems for random matrices

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Singularity of a matrix

Consider an $n \times n$ matrix M as a collection of n rows $r_i \in \mathbb{R}^n$.

$$rank(M) = \dim(span\{r_i\}_{i \in [n]}).$$

- Full rank : $rank(M) = n$
- Singular : $rank(M) < n$

A set of vectors $\{r_i\}_{i \in S}$ is a dependency if there exist non-zero coefficients $\{a_i\}_{i \in S}$ such that

$$\sum_{i \in S} a_i r_i = 0.$$

For continous random matrices, $\mathbb{P}(Singular) = 0$.

In discrete random matrices,
singularity comes from small dependencies.

$\{0,1\}$-random matrices

Let $R_n = (x_{ij})_{1\le i,j\le n}$ have iid x_{ij} r.v.

Thm [Komlós, 1967] If x_{ij} are uniform in $\{0,1\}$ then

$$\mathbb{P}(rank(R_n) < n) \le n^{-1/2}.$$

Two equal rows form a dependency.

$$\mathbb{P}(rank(R_n) < n) \ge \mathbb{P}(r_1 = r_2) \ge \left(\tfrac{1}{2}\right)^n.$$

Conjecture. [1] $\mathbb{P}(rank(R_n) < n) = \left(\tfrac{1}{2} + o(1)\right)^n.$

[1] Further results by Bourgain, Vu and Wood 2010

Symmetric $\{0,1\}$-random matrices

Let $Q_n = (x_{ij})_{1 \le i,j \le n}$ be a symmetric matrix.

- ▷ Symmetry introduces $\binom{n}{2}$ non-trivial correlations.

Thm [Costello, Tao, Vu,2006]
If x_{ij}, $i \le j$ are iid uniform in $\{0,1\}$, for any $\beta > 0$

$$\mathbb{P}(rank(Q_n) < n) = O(n^{-1/8+\beta}).$$

Rank for random matrices

For an $n \times n$ matrix M let

$$z(M) = \max(\#\{i : \text{ all entries in row } i \text{ of } M \text{ equal zero}\},$$
$$\#\{i : \text{ all entries in column } i \text{ of } M \text{ equal zero}\})$$

$$rank(M) \leq n - z(M).$$

Question. For which random matrices do we have

$$rank(M) = n - z(M)$$

with high probability?

Sparse $\{0,1\}$-random matrices

Let $Q_{n,p} = (x_{ij})_{1 \le i,j \le n}$ be symmetric with x_{ij}, $i < j$ iid Ber(p) r.v.

▷ $p < \frac{\ln n}{2n}$: 2 equal rows with prob 1-o(1)

Thm [Costello, Vu, 2006] For $p = \frac{c \ln n}{n} \le \frac{1}{2}$ and $c > \frac{1}{2}$,

$$\mathbb{P}(rank(Q_{n,p}) = n - z(Q_{n,p})) = 1 - o(1).$$

Sparse $\{0,1\}$-random matrices

Let $R_{n,p} = (x_{ij})_{1\leq i,j\leq n}$ have iid x_{ij} Ber(p) r.v.

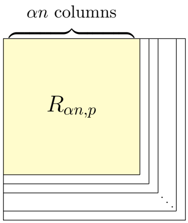
Thm [Addario-Berry,E., *CPC* 2013$^+$] For $c > \frac{1}{2}$ and $p \in (\frac{c \ln n}{n}, \frac{1}{2})$,

$$\mathbb{P}(rank(R_{n,p}) = n - z(R_{n,p})) = 1 - o(1).$$

Proof Sketch.

(Similar tecnique used by Costello, Vu)

1. Prove that whp $rank(R_{\alpha n,p}) \geq (1-\varepsilon)n$.
2. Add row/column one-at-a-time. Dependency sets will be removed by the end of the exposure.



First step (non-symmetric case)

Lemma. Let $p \in (\frac{c \ln n}{n}, \frac{1}{2})$ and $\varepsilon > 0$. There is a constant δ

$$\mathbb{P}(rank(R_{n,p}) < (1-\varepsilon)n) = O(n^{-\delta n}).$$

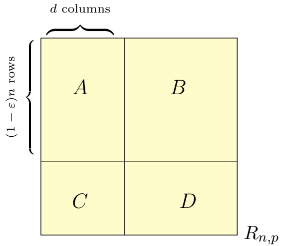
Proof. Consider a fixed $n \times n$ matrix M and suppose

$$rank(M) = d \leq (1-\varepsilon)n.$$

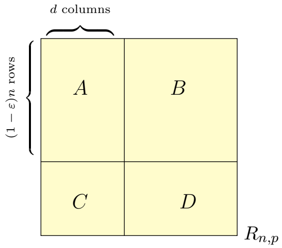
Suppose also that a reorder of columns gives

$$rank(A) = d.$$

$B \in span(A)$ so there is G such that $AG = B$. In fact, G is unique.



First step (non-symmetric case)



Likewise, D is determined by A, B, C.
Thus, given A, B, C there is a matrix F such that

$$rank(M) = d \qquad \Leftrightarrow \qquad D = M.$$

Therefore,

$$\begin{aligned}\mathbb{P}(rank(R_{n,p}) \leq (1-\varepsilon)n) &\leq \binom{n}{(1-\varepsilon)n}\mathbb{P}(rank(R_{n,p} = d|\ )\\ &\leq 2^n \max(p, 1-p)^{(\varepsilon n)^2}.\end{aligned}$$

If $p \in (\frac{c \ln n}{n}, \frac{1}{2})$, then for some $\delta > 0$

$$\mathbb{P}(rank(R_{n,p}) \leq (1-\varepsilon)n) = O(n^{-\delta n}).$$

□

Symmetric matrix process

Let $\{U_{ij}\}_{1\leq i<j\leq n}$ be independent r.v. uniform in $(0,1)$.
Let $\{Q_{n,p}\}_{p\in(0,1)}$ be the symmetric matrix-valued markov process given by

$$Q_{n,p}(i,j) = Q_{n,p}(j,i) = \mathbf{1}_{[U_{ij}\leq p]}.$$

Hitting times:

$$\tau = \inf\{p : \ rank(Q_{n,p}) = n\}$$
$$\tau_c = \inf\{p : \ z(Q_{n,p}) = 0\}.$$

▷ **Fact:** $\tau \geq \tau_c$ Proof. $rank(Q_{n,p} \leq n - z(Q_{n,p})$. □

Thm[Addario-Berry, E., *CPC* 2013+]

$$\mathbb{P}(\tau = \tau_c) \to 1, \text{ as } n \to \infty,$$

Tools

We want to understand

$$\mathbb{P}(rank(Q_{n,\tau_c}) = n)$$

- No monotonicity in the rank process.
- Structure of sparse matrices implies dependencies involve a large number of rows:

  If $S \subset [n]$ is small, then some column j has exactly one 1 whp. Then for any non-zero coefficients $\{a_i\}_{i \in S}$

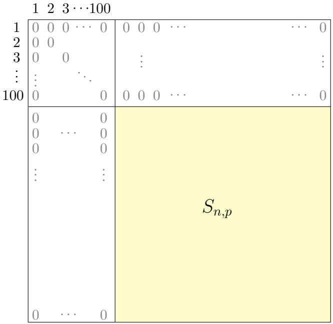
$$\sum_{i \in S} a_i r_{ij} \neq 0.$$

Proof idea

Take $z_p = z(Q_{n,p})$.

1. Set p' with $\mathbb{E}(z_{p'}) = 100$.
   It is unlikely that $\tau_c \le p'$.

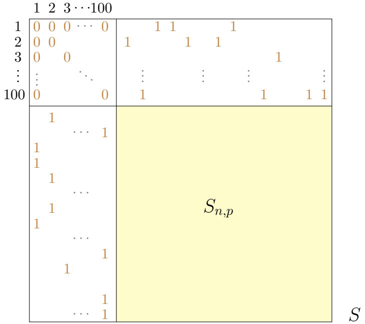
2. Condition on $z_{p'}$.
   Eg. suppose $z_{p'} = 100$, and relabel rows/col. for $Q_{n,p'}$.



$S_{n,p}$: the random submatrix of the relabelled $Q_{n,p}$.

- ▷ $S_{n,p}$ evolves indep. from zero rows/col.
- ▷ τ_c is conditionally indep. of structure of $S_{n,p}$ given $z_{p'}$.

Proof idea



- ▷ By time τ_c:
  Expected numbers of ones in rows $1, \ldots, 100$ is $O(1)$.
  Whp no dependencies in the rows $1, \ldots, 100$.

3. Condition on the values of rows $1, \ldots, 100$ at time τ_c.

We get a partially deterministic matrix S, and remains to strengthen Costello-Vu thm to S:

$$\mathbb{P}(rank(S) = n - z(S)) = 1 - o(1).$$

It is not difficult to do. :) □

Summary

In discrete random matrices,
singularity comes from small dependencies.

- All-zeros rows
- Equal rows

For the matrix process $Q_{n,p}$ with probability $1 - o(1)$:

- Equal rows are not present for $\frac{\ln n}{2n} < p < 1/2$,
- singularity disappears when the last zero row disappears.

THANKS !